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A characteristic-free example of a Lascoux resolution, and letter-place methods for intertwining numbers

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Alain Lascoux: un homme de caractère libre dont les travaux se basent en caractéristique zéro.

Abstract

An example is given to show the relationship between characteristic-free resolutions and Lascoux resolutions of Weyl modules. We also give an indication of use of letter-place methods for the calculation of intertwining numbers.

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1. Introduction

Although Alain Lascoux and I never collaborated on a mathematical paper, his ideas have been a constant influence on my students', my collaborators' and my work ever since the appearance of the first draft of his doctoral thesis [12].

His systematic use of characteristic-zero representation theory to find the resolutions of ideals generated by the minors, of any order, of generic matrices, impelled me and my students in that period to develop a characteristic-free theory of Schur and Weyl modules [4]. The attempt by Akin, Weyman and myself to use these characteristic-free methods to reproduce Lascoux' resolutions led to the realization that there were many mysteries hidden in \mathbb{Z} -forms of rational representations that had to be uncovered in order to move ahead with this project [3]. Akin and I soon discovered that the study of \mathbb{Z} -forms was intimately bound up with the resolutions of Weyl modules [1, 2], the characteristic-zero version of which Lascoux had already presented. The study of such resolutions was

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helpful in the Roberts–Weyman [13] presentation of the Hashimoto [10] example of the dependence of the Betti numbers of determinantal ideals on characteristic. Further work on these resolutions with Rota [8] led to the use of letter-place methods and place polarizations in a systematic way in this area.

In Sections 2 and 3, we will give a few examples of the way in which Lascoux' work has been incorporated into a number of the above-mentioned areas of investigation. Where possible, we will point out the similarities and differences between the classical and 'neoclassical' results. In Section 4, we give a very brief indication of how place polarization methods, Capelli identities, and resolutions come into play in the study of intertwining numbers.

2. Examples of \mathbb{Z} -forms of rational representations

In [3], we had to prove and use the rather strange fact that the complex

$$0 \rightarrow \Lambda^k \rightarrow D_1 \otimes \Lambda^{k-1} \rightarrow \dots \rightarrow D_l \otimes \Lambda^{k-l} \rightarrow \dots \rightarrow D_k \rightarrow 0, \quad (1)$$

in which the maps entail diagonalizing the exterior power and multiplying in the divided power, is exact from Λ^k up to $l = [k/2]$; that is, exact till halfway up. Thus, the kernels (or images) in those dimensions are universal \mathbb{Z} -free representations of Gl_n which, when tensored with the rationals, \mathbb{Q} , are isomorphic to the Schur modules of the hooks in the corresponding dimensions (since the Schur hooks are the kernels of the corresponding complex where the divided power, D , is replaced by the symmetric power, S). Thus, these integral representations are what Akin, Weyman and the author called \mathbb{Z} -forms of the same rational representation of Gl_n . For example, D_k and S_k are non-isomorphic \mathbb{Z} -forms of the k th symmetric power.

Another, simpler, way to construct non-isomorphic \mathbb{Z} -forms is the following:

Consider the short exact sequence

$$0 \rightarrow D_{k+2} \rightarrow D_{k+1} \otimes D_1 \rightarrow K_{(k+1,1)} \rightarrow 0$$

where $K_{(k+1,1)}$ is the Weyl module associated to the hook partition $(k+1, 1)$. If we take an integer, t , and multiply D_{k+2} by t , we get an induced exact sequence and a commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & D_{k+2} & \rightarrow & D_{k+1} \otimes D_1 & \rightarrow & K_{(k+1,1)} \rightarrow 0 \\ & & \downarrow t & & \downarrow & & \downarrow \\ 0 & \rightarrow & D_{k+2} & \rightarrow & E(t; k+1, 1) & \rightarrow & K_{(k+1,1)} \rightarrow 0, \end{array}$$

where $E(t; k+1, 1)$ stands for the cofiber product of D_{k+2} and $D_{k+1} \otimes D_1$. Each of these modules is a \mathbb{Z} -form of $D_{k+1} \otimes D_1$, but for t_1 and t_2 , two such are isomorphic if and only if $t_1 \equiv t_2 \pmod{k+2}$ (see [2]). In fact, one can easily show that $\text{Ext}_A^1(K_{(k+1,1)}, D_{k+2}) = \mathbb{Z}/(k+2)$, where A stands for the Schur algebra of appropriate degree (namely, $k+2$).

To see how such forms are related to resolutions of Weyl modules, consider the partition $(k, 2)$, the associated Weyl module, $K_{(k,2)}$, and its resolution (over the integers):

$$0 \rightarrow D_{k+2} \rightarrow \{D_{k+2} \oplus D_{k+1} \otimes D_1\} \rightarrow D_k \otimes D_2 \rightarrow K_{(k,2)} \rightarrow 0.$$

Notice that in characteristic zero, the resolution that Lascoux describes would be

$$0 \rightarrow D_{k+1} \otimes D_1 \rightarrow D_k \otimes D_2 \rightarrow K_{(k,2)} \rightarrow 0;$$

the Euler–Poincaré characteristics of the two are the same, but the first one requires more terms to take care of torsion over the integers.

The map of D_{k+2} into the indicated direct sum is the usual diagonalization into the second summand (or the place polarization ∂_{21}), but is multiplication by 2 into the first. The map from D_{k+2} to $D_k \otimes D_2$ is the second divided power of ∂_{21} or $\partial_{21}^{(2)}$. As a result, we see that the cokernel of this map is what we called above, $E(2; k+1, 1)$, and we would have to rewrite Lascoux’ resolution above as

$$0 \rightarrow E(2; k+1, 1) \rightarrow D_k \otimes D_2 \rightarrow K_{(k,2)} \rightarrow 0.$$

(This is, of course, the same as his original resolution after tensoring by \mathbb{Q} .)

The study of \mathbb{Z} -forms, especially as they emerged in our attempts to resolve determinantal ideals beyond the submaximal minors, led to the suspicion that perhaps there is no universal (i.e., integral) resolution for the minors of any order of the generic matrix. Hashimoto [10] finally took what everyone thought must be true: the Betti numbers of these determinantal ideals are independent of characteristic, and found a counter-example (the case of 2×2 minors of the 5×5 matrix). As things now stand, the resolutions of Lascoux stand as the minimal characteristic-zero universal ones; recent work of Hashimoto [11] shows that minimal universal resolutions whose terms are tilting modules, do exist.

3. Some examples of three-rowed shapes and skew hooks

One presentation of the Hashimoto counter-example mentioned above, developed by Roberts and Weyman [13], uses the characteristic-free resolution of Weyl modules in certain cases. In this section, we will look at the resolution of a three-rowed shape, and see how it differs from that constructed by Lascoux in characteristic zero. The resolutions of those for general skew hooks were given in [2] and [6, 9], while those for very special three-rowed shapes were given in [7]. The one given here also appeared in [8], but the explicit connection with the Lascoux resolution was not given there.

3.1. Lascoux and non-Lascoux resolutions for $(2, 2, 2)$

The Lascoux resolution of the Weyl module associated to the partition $(2, 2, 2)$ looks like this:

$$0 \rightarrow D_4 \otimes D_2 \otimes D_0 \rightarrow \begin{array}{c} D_3 \otimes D_3 \otimes D_0 \\ \oplus \\ D_4 \otimes D_1 \otimes D_1 \end{array} \rightarrow \begin{array}{c} D_3 \otimes D_1 \otimes D_2 \\ \oplus \\ D_2 \otimes D_3 \otimes D_1 \end{array} \rightarrow D_2 \otimes D_2 \otimes D_2 \rightarrow 0.$$

Recall that the terms of the Lascoux resolution are read off from the determinantal expansion of the Jacobi–Trudi matrix of the partition, with the position of the terms of the complex determined by the length of the permutations to which they correspond. The correspondence between the terms of the resolution above, and permutations, is

as follows:

$$D_2 \otimes D_2 \otimes D_2 \longleftrightarrow \text{identity}$$

$$D_3 \otimes D_1 \otimes D_2 \longleftrightarrow (12)$$

$$D_2 \otimes D_3 \otimes D_1 \longleftrightarrow (23)$$

$$D_3 \otimes D_3 \otimes D_0 \longleftrightarrow (123)$$

$$D_4 \otimes D_1 \otimes D_1 \longleftrightarrow (213)$$

$$D_4 \otimes D_2 \otimes D_0 \longleftrightarrow (13)$$

By contrast, the terms of the characteristic-free resolution of the same Weyl module, are:

$$X_0 = D_2 \otimes D_2 \otimes D_2;$$

$$X_1 = Z_{2,1}^{(1)} x D_3 \otimes D_1 \otimes D_2 \oplus Z_{2,1}^{(2)} x D_4 \otimes D_0 \otimes D_2 \\ \oplus Z_{3,2}^{(1)} y D_2 \otimes D_3 \otimes D_1 \oplus Z_{3,2}^{(2)} y D_2 \otimes D_4 \otimes D_0;$$

$$X_2 = Z_{2,1}^{(1)} x Z_{2,1}^{(1)} x D_4 \otimes D_0 \otimes D_2 \\ \oplus Z_{3,2}^{(1)} y Z_{2,1}^{(2)} x D_4 \otimes D_1 \otimes D_1 \oplus Z_{3,2}^{(1)} y Z_{2,1}^{(3)} x D_5 \otimes D_0 \otimes D_1 \\ \oplus Z_{3,2}^{(2)} y Z_{2,1}^{(3)} x D_5 \otimes D_1 \otimes D_0 \oplus Z_{3,2}^{(2)} y Z_{2,1}^{(4)} x D_6 \otimes D_0 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,2}^{(1)} y D_2 \otimes D_4 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z D_3 \otimes D_3 \otimes D_0;$$

$$X_3 = Z_{3,2}^{(1)} y Z_{2,1}^{(2)} x Z_{2,1}^{(1)} x D_5 \otimes D_0 \otimes D_1 \\ \oplus Z_{3,2}^{(2)} y Z_{2,1}^{(3)} x Z_{2,1}^{(1)} x D_6 \otimes D_0 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,2}^{(1)} y Z_{2,1}^{(3)} x D_5 \otimes D_1 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,2}^{(1)} y Z_{2,1}^{(4)} x D_6 \otimes D_0 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x D_4 \otimes D_2 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(2)} x D_5 \otimes D_1 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(3)} x D_6 \otimes D_2 \otimes D_0;$$

$$X_4 = Z_{3,2}^{(1)} y Z_{3,2}^{(1)} y Z_{2,1}^{(3)} x Z_{2,1}^{(1)} x D_6 \otimes D_0 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x Z_{2,1}^{(1)} x D_5 \otimes D_1 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(2)} x Z_{2,1}^{(1)} x D_6 \otimes D_0 \otimes D_0 \\ \oplus Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x Z_{2,1}^{(2)} x D_6 \otimes D_0 \otimes D_0;$$

$$X_5 = Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x Z_{2,1}^{(1)} x Z_{2,1}^{(1)} x D_6 \otimes D_0 \otimes D_0$$

where the subscripts on the X indicate the dimension in which these terms appear. The symbols $Z_{a,b}^{(t)}$ are the formal ‘polarization’ operators defined in [8], and the letters x, y, z are the separator variables also explained in that paper.

The boundary map for this complex is obtained by polarizing all the separator variables to one. When the separator x disappears between a $Z_{a,b}^{(t)}$ and elements in the tensor product of divided powers, this means $\partial_{a,b}^{(t)}$, or the place polarization operator, applied to that tensor product. The only essentially new terms that we have here are the terms that involve

$Z_{3,2}^{(1)}yZ_{3,1}^{(1)}z$, or more generally terms of the form:

$$Z_{3,2}^{(1)}yZ_{3,1}^{(1)}zZ_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xD_{2+1+|k|} \otimes D_{2+1-|k|},$$

(of which, in this example, there are not that many). For these we have to use identities of Capelli type, or some easy variants of them [8]. The boundary map on such a term sends it to

$$\begin{aligned} & Z_{3,2}^{(1)}yZ_{3,1}^{(1)}z\partial\{Z_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xD_{p+1+|k|} \otimes D_{q+1-|k|}\} \\ & \pm Z_{3,2}^{(1)}y\{Z_{3,1}^{(1)}Z_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xD_{p+1+|k|} \otimes D_{q+1-|k|}\} \\ & \mp Z_{3,2}^{(1)}Z_{3,1}^{(1)}z\{Z_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xD_{p+1+|k|} \otimes D_{q+1-|k|}\}, \end{aligned}$$

and we have to define the terms

$$Z_{3,1}^{(1)}Z_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xD_{p+1+|k|} \otimes D_{q+1-|k|}$$

and

$$Z_{3,2}^{(1)}Z_{3,1}^{(1)}z\{Z_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xD_{p+1+|k|} \otimes D_{q+1-|k|}\}$$

with $n \geq 2$.

If $n = 2$, $Z_{3,1}^{(1)}v = \partial_{3,1}(v)$, while $Z_{3,2}^{(1)}Z_{3,1}^{(1)}zv = -Z_{2,1}^{(1)}x\partial_{3,2}^{(2)}(v) + Z_{3,2}^{(2)}y\partial_{2,1}(v)$. For $n > 2$, we have

$$\begin{aligned} Z_{3,1}^{(1)}Z_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xv &= -Z_{2,1}^{(k_1+1)}xZ_{3,2}^{(1)}Z_{2,1}^{(k_2)}x \cdots xZ_{2,1}^{(k_{n-2})}xv \\ &+ \binom{k_1+k_2}{k_2-1}Z_{3,2}^{(1)}yZ_{2,1}^{(k_1+k_2+1)}x \cdots xZ_{2,1}^{(k_{n-2})}xv, \end{aligned}$$

and

$$\begin{aligned} Z_{3,2}^{(1)}Z_{3,1}^{(1)}zZ_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xv &= -Z_{2,1}^{(1)}xZ_{3,2}^{(2)}Z_{2,1}^{(k_1)}x \cdots xZ_{2,1}^{(k_{n-2})}xv \\ &+ (k_1-1)Z_{3,2}^{(2)}yZ_{2,1}^{(k_1+1)}x \cdots xZ_{2,1}^{(k_{n-2})}xv. \end{aligned}$$

3.2. Reduction of non-Lascoux to Lascoux

It is of some interest to see how we can discard the ‘excess’ terms of the characteristic-free resolution to recover that of Lascoux. For example, we want to throw away the terms

$$Z_{2,1}^{(2)}xD_4 \otimes D_0 \otimes D_2 \quad \text{and} \quad Z_{3,2}^{(2)}yD_2 \otimes D_4 \otimes D_0.$$

If we look at the image of a term $Z_{2,1}^{(2)}x \otimes v$, where $v \in D_4 \otimes D_0 \otimes D_2$, we see that it is $\partial_{2,1}^{(2)}(v) = \frac{1}{2}Z_{2,1}^{(2)}(v)$. So, the image of $Z_{2,1}^{(2)}x \otimes v$ is the same as the image of $\frac{1}{2}Z_{2,1}^{(1)}x \otimes \partial_{2,1}^{(1)}(v)$. Hence in characteristic zero we can rig up the boundary map taking this into account. Obviously the same kind of thing holds for the term $Z_{3,2}^{(2)}yD_2 \otimes D_4 \otimes D_0$ i.e., its image is the same as that of $\frac{1}{2}Z_{3,2}^{(1)}x \otimes \partial_{3,2}^{(1)}(v)$. Thus, any term that is sent by the ‘full’ boundary map into one of the ‘redundant’ terms above, should now be sent into the non-redundant term instead. For example, the following terms should now be sent as follows:

$$Z_{3,2}^{(1)}yZ_{2,1}^{(2)}xv \rightarrow Z_{3,2}^{(1)}y\partial_{2,1}^{(2)}(v) - Z_{2,1}^{(1)}x\partial_{3,1}(v) - \frac{1}{2}Z_{2,1}^{(1)}x\partial_{2,1}\partial_{3,2}(v)$$

and

$$Z_{3,2}^{(1)}yZ_{3,1}^{(1)}zv \rightarrow Z_{3,2}^{(1)}y\partial_{3,1}(v) - Z_{2,1}^{(1)}x\partial_{3,2}^{(2)}(v) - \frac{1}{2}Z_{3,2}^{(1)}y\partial_{3,2}\partial_{2,1}(v).$$

With this modification, the terms of the type

$$Z_{2,1}^{(1)}xZ_{2,1}^{(1)}xD_4 \otimes D_0 \otimes D_2$$

and

$$Z_{3,2}^{(1)}yZ_{3,2}^{(1)}yD_2 \otimes D_4 \otimes D_0$$

are automatically sent to zero. As a result, when we go to the last term that counts in this complex, we see that under the boundary map in the big complex we get

$$\begin{aligned} Z_{3,2}^{(1)}yZ_{3,1}^{(1)}zZ_{2,1}^{(1)}xv &\rightarrow Z_{3,2}^{(1)}yZ_{3,1}^{(1)}z\partial_{2,1}(v) + Z_{3,2}^{(1)}yZ_{2,1}^{(2)}x\partial_{3,2}(v) \\ &\quad - Z_{3,2}^{(1)}yZ_{3,2}^{(1)}y\partial_{2,1}^{(2)}(v) - Z_{2,1}^{(1)}xZ_{2,1}^{(1)}x\partial_{3,2}^{(2)}(v), \end{aligned}$$

so that under the modified boundary, we can simply define the boundary map on this term to be

$$Z_{3,2}^{(1)}yZ_{3,1}^{(1)}zZ_{2,1}^{(1)}xv \rightarrow Z_{3,2}^{(1)}yZ_{3,1}^{(1)}z\partial_{2,1}(v) + Z_{3,2}^{(1)}yZ_{2,1}^{(2)}x\partial_{3,2}(v).$$

Of course it remains to prove that in characteristic zero this is exact; we indicate how to do this in the next subsection.

3.3. Question of exactness

Here, we indicate more schematically how we modify the maps to take advantage of divisibility in \mathbb{Q} . Divide the terms of our big complex into the sum of those of the Lascoux complex, and the others. That is, we look at the terms of the complex as:

$$\begin{aligned} X_0 &= A_0 \\ X_1 &= A_1 \oplus B_1 \\ X_2 &= A_2 \oplus B_2 \\ X_3 &= A_3 \oplus B_3 \\ X_j &= B_j \quad \text{for } j = 4, 5, \end{aligned}$$

where the A s are the sums of the ‘Lascoux’ terms, and the B s are the sums of the others.

Let σ_1 be the map

$$B_1 \rightarrow A_1$$

defined by

$$Z_{2,1}^{(2)}xv \mapsto \frac{1}{2}Z_{2,1}^{(1)}x\partial_{2,1}(v) \quad \text{and} \quad Z_{3,2}^{(2)}yw \mapsto \frac{1}{2}Z_{3,2}^{(1)}y\partial_{3,2}(w)$$

where $v \in D_4 \otimes D_0 \otimes D_2$ and $w \in D_2 \otimes D_4 \otimes D_0$. This situation we have already discussed, but we should point out that the map σ_1 satisfies the identity:

$$\delta_{A_1A_0}\sigma_1 = \delta_{B_1B_0},$$

where by $\delta_{A_1 A_0}$ we mean the component of the boundary of the fat complex that carries A_1 to A_0 . We will use notation $\delta_{A_{i+1} A_i}$, $\delta_{A_{i+1} B_i}$ etc. in the same way. We define

$$\partial_1 : A_1 \rightarrow A_0$$

as

$$\partial_1 = \delta_{A_1 A_0}.$$

But now we are in position to define

$$\partial_2 : A_2 \rightarrow A_1$$

by

$$\partial_2 = \delta_{A_2 A_1} + \sigma_1 \delta_{A_2 B_1}.$$

We get immediately

Fact 3.1. The composition $\partial_1 \partial_2 = 0$.

Proof. We have

$$\begin{aligned} \partial_1 \partial_2(a) &= \delta_{A_1 A_0} \{ \delta_{A_2 A_1}(a) + \sigma_1 \delta_{A_2 B_1}(a) \} \\ &= \delta_{A_1 A_0} \delta_{A_2 A_1}(a) + \delta_{A_1 A_0} \sigma_1 \delta_{A_2 B_1}(a). \end{aligned}$$

But since $\delta_{A_1 A_0} \sigma_1 = \delta_{B_1 B_0}$, we see that

$$\delta_{A_1 A_0} \sigma_1 \delta_{A_2 B_1}(a) = \delta_{B_1 B_0} \delta_{A_2 B_1}(a),$$

so

$$\delta_{A_1 A_0} \delta_{A_2 A_1}(a) + \delta_{A_1 A_0} \sigma_1 \delta_{A_2 B_1}(a) = \delta_{A_1 A_0} \delta_{A_2 A_1}(a) + \delta_{B_1 B_0} \delta_{A_2 B_1}(a).$$

But this is zero since it is the boundary of the fat complex applied to a . \square

Before we can prove exactness at A_1 , we have to define a map

$$\sigma_2 : B_2 \rightarrow A_2$$

such that

$$\delta_{B_2 A_1} + \sigma_1 \delta_{B_2 B_1} = \{ \delta_{A_2 A_1} + \sigma_1 \delta_{A_2 B_1} \} \sigma_2. \quad (\text{C}_2)$$

We define this map as follows:

$$\begin{aligned} Z_{2,1}^{(1)} x Z_{2,1}^{(1)} x v &\longmapsto 0 \\ Z_{3,2}^{(1)} y Z_{3,2}^{(1)} y v &\longmapsto 0 \\ Z_{3,2}^{(1)} y Z_{2,1}^{(3)} x v &\longmapsto \frac{1}{3} \{ Z_{3,2}^{(1)} y Z_{2,1}^{(2)} x \partial_{2,1}(v) \} \\ Z_{3,2}^{(2)} y Z_{2,1}^{(3)} x v &\longmapsto \frac{1}{3} \{ Z_{3,2}^{(1)} y Z_{2,1}^{(2)} x \partial_{3,1}(v) - Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z \partial_{2,1}^{(2)}(v) \} \\ Z_{3,2}^{(2)} y Z_{2,1}^{(4)} x v &\longmapsto -\frac{1}{2} \{ Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z \partial_{2,1}^{(3)}(v) \} \end{aligned}$$

where the element v is in the appropriate tensor product of divided powers as given in the description of the fat complex.

Fact 3.2. The map σ_2 defined above satisfies the condition (C₂).

Proof. Trivial. \square

Fact 3.3. We have exactness at A_1 .

Proof. This is just a diagram chase. \square

Using σ_2 we can now also define

$$\partial_3 : A_3 \rightarrow A_2$$

by

$$\partial_3 = \delta_{A_3 A_2} + \sigma_2 \delta_{A_3 B_2}.$$

Not too surprisingly, we have

Fact 3.4. $\partial_2 \partial_3 = 0$.

Of course, what we need now is a map $\sigma_3 : B_3 \rightarrow A_3$ similar to the σ 's above, i.e., satisfying

$$\delta_{B_3 A_2} + \sigma_2 \delta_{B_3 B_2} = \{\delta_{A_3 A_2} + \sigma_2 \delta_{A_3 B_2}\} \sigma_3. \quad (\text{C}_3)$$

We define such a σ_3 as follows:

$$\begin{aligned} Z_{3,2}^{(1)} y Z_{2,1}^{(2)} x Z_{2,1}^{(1)} x v &\mapsto 0 \\ Z_{3,2}^{(2)} y Z_{2,1}^{(3)} x Z_{2,1}^{(1)} x v &\mapsto \frac{1}{3} \{ Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x \partial_{2,1}^{(2)}(v) \} \\ Z_{3,2}^{(1)} y Z_{3,2}^{(1)} y Z_{2,1}^{(3)} x v &\mapsto -\frac{1}{3} \{ Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x \partial_{2,1}^{(1)}(v) \} \\ Z_{3,2}^{(1)} y Z_{3,2}^{(1)} y Z_{2,1}^{(4)} x v &\mapsto -\frac{1}{3} \{ Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x \partial_{2,1}^{(2)}(v) \} \\ Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x v &\mapsto \frac{1}{3} \{ Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(1)} x \partial_{2,1}^{(1)}(v) \} \\ Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(3)} x v &\mapsto 0 \end{aligned}$$

where the element v again is taken in the appropriate tensor product of divided powers as described earlier.

Fact 3.5. The map σ_3 defined above satisfies the condition (C₃).

Proof. Trivial. \square

We have yet to prove that we have exactness at A_2 and that ∂_3 is a monomorphism. But first we explicitly define the boundary maps in the complex

$$0 \rightarrow A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

The map ∂_1 is clear: it is just the operation of the indicated polarization operators on the argument. The map ∂_2 is defined as:

$$\begin{aligned} \partial_2 \left(Z_{3,2}^{(1)} y Z_{2,1}^{(2)} x v \right) &= Z_{3,2}^{(1)} y \partial_{2,1}^{(2)}(v) - Z_{2,1}^{(1)} x \partial_{3,1}^{(1)}(v) - \frac{1}{2} Z_{2,1}^{(1)} x \partial_{2,1}^{(1)} \partial_{3,2}^{(1)}(v); \\ \partial_2 \left(Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z v \right) &= Z_{3,2}^{(1)} y \partial_{3,1}^{(1)}(v) - \frac{1}{2} Z_{3,2}^{(1)} y \partial_{3,2}^{(1)} \partial_{2,1}^{(1)}(v) + Z_{2,1}^{(1)} x \partial_{3,2}^{(2)}(v). \end{aligned}$$

map $\text{Hom}_A(\square, K_\mu)$ (see [5]), and since the domain and range of \square consists of sums of tensor products of divided powers, this amounts to calculating a basis for certain weight submodules of K_μ , and then making explicit the entries of the matrix of the map above with respect to these bases.

The weight submodule of K_μ corresponding to weight λ is parametrized by three parameters $3_1, 4_1, 4_2$, where n_i indicates the number of n 's in row i . The homomorphism corresponding to these parameters, $\alpha^{3_1, 4_1, 4_2}$, may be described as the composition of place polarizations:

$$\alpha^{3_1, 4_1, 4_2} = \partial_{34}^{(d-4_1-4_2)} \partial_{24}^{(4_2)} \partial_{14}^{(4_1)} \partial_{23}^{(d-3_1-4_1-4_2)} \partial_{13}^{(3_1)} \partial_{12}^{(d-3_1-4_1)}.$$

The only restrictions on the parameters are that

$$3_1 + 4_1 + 4_2 \leq d$$

and

$$4_2 \leq s_2.$$

We see immediately that:

$$\begin{aligned} 2_1 &= d - 3_1 - 4_1 & 2_2 &= s_3 + s_2 + 3_1 + 4_1 \\ 3_2 &= d - 3_1 - 4_1 - 4_2 & 3_3 &= s_3 + 4_1 + 4_2 \\ 4_3 &= d - 4_1 - 4_2. \end{aligned}$$

The idea of using the place polarizations is that composing the alphas with \square just amounts to composing the given composite of place polarizations with either $\partial_{21}^{(l)}$, $\partial_{32}^{(l)}$ or $\partial_{43}^{(l)}$ for some positive integer l . Since all of these polarizations are taking place at the level of the generators of the Weyl modules involved, that is, in the tensor product of appropriate divided powers, we want to throw away those terms that are obviously in the kernel of the Weyl map defining K_μ . But those are any terms which are composites of place polarizations that have either a $\partial_{21}^{(l)}$ or $\partial_{32}^{(l)}$ at the extreme left end of the composites. Thus when we take an alpha and compose it with one of the $\partial_{21}^{(l)}$, $\partial_{32}^{(l)}$ or $\partial_{43}^{(l)}$, we want to use the Heisenberg–Weyl identities to bring these all the way to the left, if we can.

That is the general strategy behind the simplification, and there is some evidence that this can be carried out. At least, this is a program on which a group of us are currently engaged.¹ As a more detailed description of the fragments we have so far obtained would take more space than would be justified by the results, we will forgo giving one here.

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¹ D. Flores, R. Sanchez and the author are presently the members of this group.

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